On Intersection Representations and Clique Partitions of Graphs*

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Abstract

A multifamily set representation of a finite simple graph G is a multifamily $\mathcal{F}$ of sets (not necessarily distinct) for which each set represents a vertex in G and two sets in $\mathcal{F}$ intersects if and only if the two corresponding vertices are adjacent. For a graph G, an *edge clique covering* (*edge clique partition*, respectively) $\mathcal{Q}$ is a set of cliques for which every edge is contained in *at least* (*exactly*, respectively) one member of $\mathcal{Q}$. In 1966, P. Erdös, A. Goodman, and L. Pósa (The representation of a graph by set intersections, *Canadian J. Math.*, **18**, pp.106-112) pointed out that for a graph there is a one-to-one correspondence between multifamily set representations $\mathcal{F}$ and clique coverings $\mathcal{Q}$ for the edge set. Furthermore, for a graph one may similarly have a one-to-one correspondence between particular multifamily set representations with intersection size at most one and clique partitions of the edge set. In 1990, S. McGuinness and R. Rees (On the number of distinct minimal clique partitions and clique covers of a line graph, *Discrete Math.* **83** (1990) 49-62.) calculated the number of distinct

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clique partitions for line graphs. In this paper, we study the set representations of graphs corresponding to edge clique partitions in various senses, namely family representations of *distinct* sets, antichain representations of *mutually exclusive* sets, and uniform representations of sets with the *same cardinality*. Among others, we completely determine the number of distinct family representations and the number of antichain representations of line graphs.

1 Background and Introduction

By a *multigraph* $M = (V(M), E(M), q)$ we mean a triple consisting of a set $V(M)$ of *vertices*, a set $E(M)$ of *edges*, and a function q defined in the following manner. For each unordered pair $\{u, v\} \subset V(M)$, let $q(u, v)$ be the number of *parallel edges* joining u with v, and we call it the *multiplicity*. If $q(u, v) \neq 0$, then we say that $\{u, v\}$ is an *edge* of M. In this paper unless otherwise stated we consider only *finite, undirected, simple* graphs. That is, $q(u, v) \leq 1$ for every $\{u, v\} \subset V(M)$ and $q(u, u) = 0$ for every $u \in V(M)$. We simply call them graphs instead of multigraphs throughout this article. For a subset $S \subseteq V(M)$, $\langle S \rangle_V$ denotes the *subgraph induced by* S. For a vertex v in a graph G, let $d_G(v)$ or $d(v)$ denote the *degree* of v in G which is the number of neighbors of v in G. For other terminology we do not define here please refer to [16].

Let $\mathcal{F} = \{S_1, ..., S_p\}$ be a multifamily of nonempty subsets of a finite nonempty set X, where $S_1, ..., S_p$ might not be distinct. $\mathbf{S}(\mathcal{F})$ denotes the union of sets in $\mathcal{F}$. The *intersection multigraph* of $\mathcal{F}$, denoted $\Omega(\mathcal{F})$, is defined by $V(\Omega(\mathcal{F})) = \mathcal{F}$, with $|S_i \cap S_j| = q(S_i, S_j)$ whenever $i \neq j$.

We say that a multigraph M is (isomorphic to) an intersection multigraph on $\mathcal{F}$ if there exists a multifamily $\mathcal{F}$ such that $M \cong \Omega(\mathcal{F})$. In this case, we also say that $\mathcal{F}$ is a *multifamily representation* of the multigraph M. It is easy to see that every multigraph is isomorphic to an intersection multigraph on some multifamily, therefore one may define the *multifamily intersection number*, denoted $\omega_m(M)$, of a given multigraph M to be the minimum cardinality of a set X such that M is isomorphic to an intersection multigraph on a multifamily $\mathcal{F}$ of subsets of X. In this case we also say that $\mathcal{F}$ is a *minimum multifamily representation* of M.

If a multigraph M is isomorphic to an intersection multigraph $\Omega(\mathcal{F})$, then we have the following additional conventions:

1. $\mathcal{F}$ is called a **family representation** of the multigraph M if $\mathcal{F}$ is a family of **distinct** subsets of X;

2. $\mathcal{F}$ is called an **antichain representation** of M if $\mathcal{F}$ is an antichain with respect to set inclusions, that is, any two sets in $\mathcal{F}$ are mutually exclusive.

3. $\mathcal{F}$ is called an **uniform representation** of M if $\mathcal{F}$ is an uniform family of distinct sets with the same cardinality.

It is not hard to see that every multigraph is isomorphic to an intersection multigraph $\Omega(\mathcal{F})$, where $\mathcal{F}$ can be required to be either a family representation, an antichain representation, or an uniform representation. Therefore it makes sense to define the following notions.

1. The **family intersection number** of M, denoted $\omega_f(M)$, is the minimum cardinality of $\mathbf{S}(\mathcal{F})$ for which M has a family representation $\mathcal{F}$.

2. The **antichain intersection number** of M, denoted $\omega_a(M)$, is the minimum cardinality of $\mathbf{S}(\mathcal{F})$ for which M has an antichain representation $\mathcal{F}$.

3. The **uniform intersection number** of M, denoted $\omega_u(M)$, is the minimum cardinality of $\mathbf{S}(\mathcal{F})$ for which M has an uniform representation $\mathcal{F}$.

Remark 1.1. *Clearly every uniform representation is an antichain representation, every antichain representation is a family representation, and every family representation is a multifamily representation. Hence* $\omega_m(M) \leq \omega_f(M) \leq \omega_a(M) \leq \omega_u(M)$*. Note that given a multifamily representation* $\{S_v \mid v \in V(M)\}$ *of* M *and a vertex subset* $S \subseteq V(M)$*, then* $\{S_v \mid v \in S\}$ *form a multifamily representation of* $\langle S\rangle_V$*. Thus we know that* $\omega_m(M) \geq \omega_m(\langle S\rangle_V)$ *for any* $S \subseteq V(M)$*. Similarly,* $\omega_f(M) \geq \omega_f(\langle S\rangle_V)$*,* $\omega_a(M) \geq \omega_a(\langle S\rangle_V)$*, and* $\omega_u(M) \geq \omega_u(\langle S\rangle_V)$ *for any* $S \subseteq V(M)$*.*

We may define the notion of uniqueness of representations of intersection multigraphs. A multigraph M is said to be **uniquely intersectable with respect to multifamilies (u.i.m.)** if given a set X with $|X| = \omega_m(M)$ and for any two families α and β of subsets of X such that α and β are both multifamily representations of M, then β can be obtained from α by a permutation of elements of X. Similarly M is **uniquely intersectable with respect to families (u.i.f.) uniquely intersectable with respect to antichains (u.i.a.)**, and **uniquely intersectable with respect to uniform families (u.i.u.)** are also defined.

Given a graph G, $Q \subseteq V(G)$ is said to be a *clique* of G if every pair of distinct vertices u, v in Q are adjacent. A *clique partition* $\mathcal{Q}$ of a graph is a

set of cliques such that every pair of distinct vertices u, v in $V(G)$ appears in exactly one clique in $\mathcal{Q}$ and for each *isolated vertex*, we need to use at least one *trivial clique* with only one vertex in $\mathcal{Q}$ to cover it. The minimum cardinality of a clique partition of G is called the *clique partition number* of G, and is denoted by $cp(G)$. We refer to a clique partition of G with the cardinality $cp(G)$ as a *minimum clique partition* of G. Note that a clique partition $\mathcal{Q}$ of a graph G gives rise to a clique partition of $G-v$ by deleting the vertex v from each clique in $\mathcal{Q}$. Thus $cp(G)$ is not less than the clique partition number of any induced subgraph of G.

In the following, we describe the correspondence between the multifamily set representations and clique partitions of the edge set for a multigraph $M = (V(M), E(M), q)$.

We first construct a clique partition

$$\mathcal{Q} = \{Q_1, ..., Q_p\}$$

of M, then with each clique Q_k we associate an element e_k and with each vertex v_α we associate a set $S_\mathcal{Q}(v_\alpha)$ of elements e_k, where

$$e_k \in S_\mathcal{Q}(v_\alpha) \Leftrightarrow v_\alpha \in Q_k,$$

that is, $S_\mathcal{Q}(v_\alpha)$ is the collection of those elements for which the corresponding cliques contains v_α. Thus we obtain

$$\mathcal{F}(\mathcal{Q}) \equiv \{S_\mathcal{Q}(v) : v \in V(M)\}.$$

Then clearly

$$\mathbf{S}(\mathcal{F}(\mathcal{Q})) \equiv \bigcup_{v \in V(M)} S_\mathcal{Q}(v)$$

contains p elements, and

$$|S_\mathcal{Q}(v_\alpha) \cap S_\mathcal{Q}(v_\beta)| = q(v_\alpha, v_\beta),$$

since there is exactly $q(v_\alpha, v_\beta)$ cliques simultaneously containing the two vertices v_α and v_β. Thus we have constructed a multifamily representation

$$\mathcal{F}(\mathcal{Q}) = \{S_\mathcal{Q}(v) : v \in V(M)\}$$

from the clique partition $\mathcal{Q}$ of M, where

$$|\mathbf{S}(\mathcal{F}(\mathcal{Q}))| \equiv |\bigcup_{v \in V(M)} S_\mathcal{Q}(v)| = p = |\mathcal{Q}|.$$

Conversely, given a multifamily representation $\mathcal{F} = \{S_1, ...S_n\}$ of G with vertex set $V(G) = \{v_1, ..., v_n\}$, where S_α correspond to the set attaching to v_α, then we can also construct a clique partition of G by the following way.

Let

$$\mathbf{S}(\mathcal{F}) \equiv \bigcup_{\alpha=1}^{n} S_\alpha = \{e_1, ..., e_p\}.$$

For each fixed e_k in $\mathbf{S}(\mathcal{F})$ we form a clique $Q_{\mathcal{F}}(e_k)$ using those vertices v_α such that the set S_α attaching to it contains e_k. Clearly each $Q_{\mathcal{F}}(e_k)$ is indeed a clique of G. Thus we obtain

$$\mathcal{Q}(\mathcal{F}) = \{Q_{\mathcal{F}}(e_1), ..., Q_{\mathcal{F}}(e_p)\}.$$

and

$$q(v_\alpha, v_\beta) = |S_\alpha \cap S_\beta|$$
$$= \text{the number of cliques in } \mathcal{Q}(\mathcal{F}) \text{ simultaneously containing } v_\alpha \text{ and } v_\beta,$$

since each element in S_α exactly represent a clique in $\mathcal{Q}(\mathcal{F})$ containing v_α. Thus we have constructed a clique partition $\mathcal{Q}(\mathcal{F})$ of M from the multifamily representation $\mathcal{F}$ of M, where

$$|\mathcal{Q}(\mathcal{F})| = p = |\bigcup_{\alpha=1}^{n} S_\alpha| \equiv |\mathbf{S}(\mathcal{F})|.$$

From above, we may treat a graph G as a special case of a multigraph and hence we have the correspondence for G, and in particular $\omega_m(G) = cp(G)$.

Intersection graphs and set representations of graphs were first introduced and studied by E. Szpilrajn-Marczewski [13] in 1945, and P. Erdös, A. Goodman, and L. Pósa [8] in 1966. The set representations in various senses such as family representations, antichain representations, uniform representations, and the associated uniqueness properties were studied in literatures over the decades [1, 6, 7, 14, 15]. In 1997, Bylka and Komar [6] tried to characterize the line graphs with a unique multifamily representation, in other words, to characterize all line graphs with a unique edge clique partition. They studied the problem and solved for the characterization with one case unsettled. In fact by S. McGuinness and R. Rees's results in [10] (Theorem 4.1 in this paper), one may solve the remaining case and complete the whole characterization of the uniqueness of the multifamily representation for line graphs, which was already pointed out in T.-M. Wang's Ph.D. thesis [15] in 1997.

In this paper, we study and completely classify the family and antichain representations for line graphs, from which the associated uniqueness results

are derived. In section 2 and section 3, we describe the relationship between the theory of finite projective plane and edge clique partition of complete graphs, then calculate the intersection numbers and classify minimum set representations of a complete graph K_n in various senses. In following section 4 and section 5, we completely determine the number of distinct minimum family representations and minimum antichain representations of a line graph, respectively. In section 6 we conclude with certain future research directions.

2 Edge Clique Partitions and Finite Projective Spaces

A *finite linear sapce* $\Gamma = (\mathcal{P}, \mathcal{L})$ is a system consisting of a finite set $\mathcal{P}$ of n *points* and a set $\mathcal{L}$ of *lines* satisfying the following axioms.

(L1) Any line has at least two points.

(L2) Two points are on precisely one line.

(L3) Any line has at most $n-1$ points.

If a space satisfy **(L1)** and **(L2)** but not **(L3)**, then clearly this space contain a unique line. This type of spaces is referred to as *trivial linear space*.

Suppose that $n \geq 3$. Let $\mathcal{Q}$ be a clique partition of the complete graph K_n such that each member of $\mathcal{Q}$ has at least 2 and no more than $n-1$ vertices. Let $\Gamma(\mathcal{Q})$ be the system whose set of points is the vertex set of K_n, and whose lines are the members of $\mathcal{Q}$. Incidence is defined as following. A points v is incident with a line Q if v is a vertex of Q. Then $\Gamma(\mathcal{Q})$ is a finite linear space. Conversely, if Γ is a finite linear space on n points, then there is a clique partition $\mathcal{Q}$ of K_n such that $\Gamma = \Gamma(\mathcal{Q})$, where each member of $\mathcal{Q}$ has at least 2 and no more than $n-1$ vertices.

Thus there is a one-one correspondence between all clique partitions of the complete graph K_n by cliques with cardinality at least 2 and at most $n-1$, and all finite linear spaces with n points.

A *projective plane* is a finite linear spaces Π satisfying further the following two axioms.

(P1) Any two distinct lines have a point in common.

(P2) There are four points, no three of which are on a common line.

Suppose that Π is a projective plane with a finite number n of points and a finite number l of lines. Then it is probative that for some integer $k \geq 2$,

$n = l = k^2 + k + 1$, and Π has point and line regularity $k + 1$, where each point is on exactly $k + 1$ lines and each line contains exactly $k + 1$ points. We call such a number k the order of the projective plane. Besides, any two lines in a projective plane intersect on a common point, or paraphrased into terms of clique partition, any two cliques intersect on a common vertex.

The smallest projective plane has order $k = 2$, which is the *Fano Plane*, as illustrated in Figure 1. Note that the segments (straight or round) passing through $\{a, b, c\}, \{c, d, e\}, \{a, f, e\}, \{a, g, d\}, \{b, g, e\}, \{f, g, c\}, \{b, d, f\}$ respectively stand for seven lines.

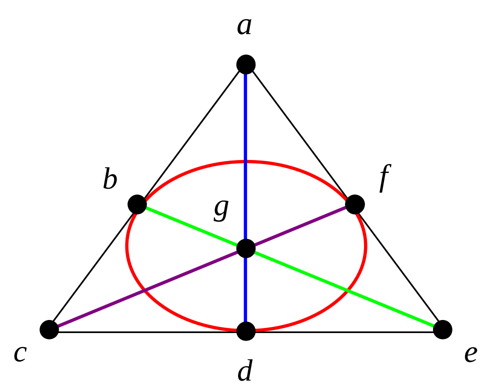

Figure 1: Fano Plane

It is well known that there are unique projective planes of orders 2, 3, 4, 5, 7, and 8 respectively while there are none of order 6 and 10 [11]. There are at least 4 non-isomorphic projective planes of order 9, but no one yet know the exactly number. It is also known that one may construct a finite projective plane of order p^k, where p is prime, based on the theory of finite fields.

In 1948, de Bruijn and Erdös proved a theorem about linear space which we paraphrase in terms of clique partition as follows.

Theorem 2.1. (de Bruijn and Erdös[5], 1948) *If $\mathcal{Q}$ with $|\mathcal{Q}| > 1$ is a clique partition of K_n with $n \geq 3$, no one of whose members is a trivial clique, that is, the clique consisting of one single vertex, then $|\mathcal{Q}| \geq n$, with equality if and only if*

(a) *$\mathcal{Q}$ consists of one clique on $n - 1$ vertices and $n - 1$ copies of K_2,*

or

(b) *The finite linear space corresponding to $\mathcal{Q}$ is a projective plane.*

Note that the linear spaces corresponding to the class of clique partitions in (a) are traditionally referred to as *near-pencil* in finite linear space theory. The example for the above Theorem 2.1 can refer to the one that the nontrivial clique partitions of K_7 use at least 7 cliques, and the extremal cases happen when they consist of either six $K_2's$ with one K_6, or seven $K_3's$ corresponding to the Fano plane.

3 Intersection Representations of Complete Graphs

In this section, we calculate the intersection numbers and study the uniqueness of the minimum representations for complete graphs K_n in various senses. We start from the family representations.

Theorem 3.1. *Let K_n be a complete graph on n vertices, $n \geq 3$.*

1. $\omega_f(K_n) = n$.
2. *K_n is not u.i.f.*

Proof. For any complete graph K_n with $n \geq 3$, we can always construct a family representation by the following method. Take an element, say e_1 common to the representation sets of all vertices. Then attach elements $e_2, ..., e_{n-1}$ to some $n-1$ vertices of the n vertices, respectively. On the other hand, there cannot exist a representation $\mathcal{F}$ of K_n with $|\mathbf{S}(\mathcal{F})| \leq n-1$, for otherwise we can first delete all elements in $\mathbf{S}(\mathcal{F})$ that appear in the representation set of only one vertex, which we would referred to as *monopolized element* in the rest of this paper, from the representation sets of all vertices and say the resulting representation $\mathcal{F}'$. Note that $\mathcal{F}'$ is a multifamily representation of K_n, since monopolized elements have nothing to do with multifamily representation of a multigraph. Now we take $\mathcal{Q}(\mathcal{F}')$. Note that $|\mathcal{Q}(\mathcal{F}')| \leq n-1$. Clearly $\mathcal{Q}(\mathcal{F}')$ is a clique partition of K_n containing no trivial clique. By theorem 2.1 and the fact that $|\mathcal{Q}(\mathcal{F}')| \leq n-1$, we know that $|\mathcal{Q}(\mathcal{F}')| = 1$, that is, $\mathcal{Q}(\mathcal{F}')$ consists of only one clique, containing all n vertices of K_n. But clearly we cannot recover $\mathcal{F}$ from $\mathcal{F}(\mathcal{Q}(\mathcal{F}')) = \mathcal{F}'$ by adding monopolized elements to the members of $\mathcal{F}'$, since $\mathcal{F}$ is a representation of K_n with $|\mathbf{S}(\mathcal{F})| \leq n-1$, a contradiction. From above we know that $\omega(K_n) = n$.

Now we investigate the uniqueness of K_n's representation. Assume a representation $\mathcal{F}$ of K_n with $|\mathbf{S}(\mathcal{F})| = n$. Delete all monopolized elements in $\mathbf{S}(\mathcal{F})$ from the representation sets of all vertices, say the resulting representation $\mathcal{F}'$, and then take $\mathcal{Q}(\mathcal{F}')$. Now $|\mathcal{Q}(\mathcal{F}')| \leq n$. Clearly $\mathcal{Q}(\mathcal{F}')$ is a clique partition of K_n containing no trivial clique. By theorem 2.1 and

$|\mathcal{Q}(\mathcal{F}')| \leq n$, we know that $\mathcal{Q}(\mathcal{F}')$ consists of only one clique, or is a near-pencil or projective plane. If $\mathcal{Q}(\mathcal{F}')$ is a near-pencil or projective plane, then $|\mathcal{Q}(\mathcal{F}')| = n$ and thus it is clear that in these two cases we had never deleted any monopolized element from the representation set of any vertex when we proceed from $\mathcal{F}$ to $\mathcal{F}'$. Thus in these two cases, the original representation $\mathcal{F}$ is just $\mathcal{F}$(near-pencil) or $\mathcal{F}$(projective plane), note that here and in the following we use these two terminologies "near-pencil" and "projective plane" to stand for their corresponding clique partitions, respectively. Clearly these two representations indeed have their constituting sets pairwise distinct.

For the remaining case, $\mathcal{Q}(\mathcal{F}')$ consists of only one clique. Thus in this case we must had deleted $n-1$ monopolized elements in proceeding from $\mathcal{F}$ to $\mathcal{F}'$. And clearly all constituting sets of $\mathcal{F}$ has a common element, say e_1, and some $n-1$ constituting sets of $\mathcal{F}$ have monopolized elements, say $e_2, ..., e_{n-1}$, respectively.

Thus in above we have proved that every complete graph K_n with $n \geq 3$ has intersection number n and has three manners for forming its minimum representations. Note that the practicability of the one manner derived from projective plane depends on whether or not $n = k^2 + k + 1$ for some $k \geq 2$ and there exists projective plane of order k. $\square$

Then we investigate the minimum antichain representations of complete graphs.

Theorem 3.2. *Let K_n be a complete graph on n vertices, where $n \geq 3$.*

1. $\omega_a(K_n) = n$.
2. *K_n is not u.i.a.*

Proof. Because $\mathcal{F}$(near-pencil) itself is a antichain representation of K_n making use of n elements, we know that $\omega_a(K_n) \leq n$. Assuming an antichain representation $\mathcal{F}$ of K_n with $|\mathbf{S}(\mathcal{F})| \leq n$. Delete all monopolized elements in $\mathbf{S}(\mathcal{F})$ from the representation set of all vertices, say the resulting representation $\mathcal{F}'$, and then take $\mathcal{Q}(\mathcal{F}')$. Now $|\mathcal{Q}(\mathcal{F}')| \leq n$ and $\mathcal{Q}(\mathcal{F}')$ is a clique partition of K_n with no trivial clique. By theorem 2.1, we know that $\mathcal{Q}(\mathcal{F}')$ have only one member, or is a near-pencil, or projective plane. Clearly $\mathcal{F}$(near-pencil) and $\mathcal{F}$(projective plane) are both antichain representation. As for the remaining case that $\mathcal{Q}(\mathcal{F}')$ have only one member, we cannot recover $\mathcal{F}$ from $\mathcal{F}(\mathcal{Q}(\mathcal{F}')) = \mathcal{F}'$ by adding monopolized elements to the members of $\mathcal{F}'$, since $\mathcal{F}$ is an antichain representation of K_n with $|\mathbf{S}(\mathcal{F})| \leq n$.

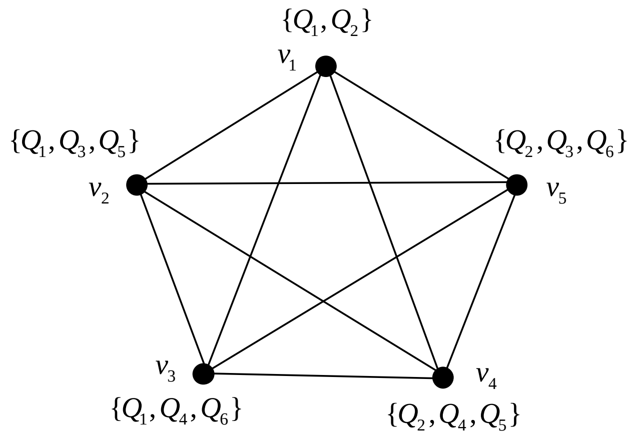

Figure 2: Finite linear space with 5 points and 6 lines

Thus we have proved that every complete graph K_n with $n \geq 3$ has antichain intersection number n and has two manners for forming its minimum antichain representations with the one manner derived from projective plane being provisory upon the existence of projective plane of appropriate order. $\square$

As for the investigation of the minimum uniform representations of complete graphs, we shall refer to the following theorem due to W. G. Bridges.

Theorem 3.3. (W. G. Bridges[4], 1972) *Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a finite linear space with $n \neq 5$ points and l lines. Then $l = n + 1$ if and only if Γ is a projective plane with one point removed from $\mathcal{P}$ and every line of $\mathcal{L}$. As for the case of $n = 5$, please see Figure 2.*

Theorem 3.4. *We have the following cases for the uniform representations of complete graphs:*

1. *for $n = 3$, K_n has uniform intersection number n and has only one manner to form its minimum uniform representations;*
2. *for $n \geq 4$, where $n = k^2 + k + 1$ for some $k \geq 2$ so that there exists projective plane of order k, K_n has uniform intersection number n and has only one manner to form its minimum uniform representations;*
3. *for $n \geq 4$ where $n = k^2 + k$ for some $k \geq 2$ so that there exists projective plane of order k, K_n has uniform intersection number $n + 1$ and has two manners to form its minimum uniform representations; and*

4. *for $n \geq 4$ where $n \neq k^2 + k + 1$ and $n \neq k^2 + k$ for all $k \geq 2$, K_n has uniform intersection number $n + 1$ and has only one manner to form its minimum uniform representations.*

Proof. Assume an uniform representation $\mathcal{F}$ of K_n with $|\mathbf{S}(\mathcal{F})| \leq n$. Delete all monopolized elements in $\mathbf{S}(\mathcal{F})$ from the representation sets of all vertices, say the resulting representation $\mathcal{F}'$, and then take $\mathcal{Q}(\mathcal{F}')$. Now $|\mathcal{Q}(\mathcal{F}')| \leq n$ and $\mathcal{Q}(\mathcal{F}')$ is a clique partition of K_n with no trivial clique. By Theorem 2.1, we know that $\mathcal{Q}(\mathcal{F}')$ have only one member, or is a near-pencil, or a projective plane. Clearly $\mathcal{F}$(projective plane) is an uniform representation in its own right, while we cannot recover an uniform representation $\mathcal{F}$, with $|\mathbf{S}(\mathcal{F})| \leq n$, of K_n with $n \geq 3$ from $\mathcal{F}$(near-pencil) by adding monopolized elements to the members of it except possibly $n = 3$. And for the remaining case, that is, $\mathcal{Q}(\mathcal{F}')$ has only one clique, we also cannot recover $\mathcal{F}$ from $\mathcal{F}(\mathcal{Q}(\mathcal{F}'))$.

Thus whenever $n \geq 4$, we have that $\omega_u(K_n) = n$ and K_n is u.i.u. if and only if $n = k^2 + k + 1$ for some $k \geq 2$ and there exists projective plane of order k.

In case that $4 \leq n \neq k^2 + k + 1$ or there exists no projective plane of order k, since we can always form an uniform representation $\mathcal{F}$ of K_n with $|\mathbf{S}(\mathcal{F})| = n + 1$ by first adopting an element common to the representation set of all vertices and then for the representation set of each vertex attaching a monopolized element to it. Thus for this case we have $\omega_u(K_n) = n + 1$. Now given an uniform representation $\mathcal{F}$ of K_n with $|\mathbf{S}(\mathcal{F})| \leq n + 1$, first we delete all monopolized elements of $\mathbf{S}(\mathcal{F})$ from the representation set of each vertex resulting in another representation, say $\mathcal{F}'$, and then take $\mathcal{Q}(\mathcal{F}')$. Now $|\mathcal{Q}(\mathcal{F}')| \leq n + 1$. By theorem 2.1 and 3.3, (note that we have assumed that $n \geq 4$ and there exists no projective plane of appropriate order) and the fact that we cannot recover $\mathcal{F}$ from $\mathcal{F}$(near-pencil with $n \geq 4$ vertices) or $\mathcal{F}$(the clique partition as in Figure 5), we know that $\mathcal{Q}(\mathcal{F}')$ either consists of only one clique, or is a projective plane with one vertex deleted. The corresponding representation of the latter is an uniform representation in its own right and we can easily recover $\mathcal{F}$ from the corresponding representation set of the former by returning monopolized element to each member of $\mathcal{F}$. $\square$

4 Family Representations of Line Graphs

The *line graph* of a graph G, which we assume to be finite, undirected and simple in this paper, written as $L(G)$, is the graph whose vertices are the edges of G, with its two vertices adjacent if and only if the two edges in G corresponding to these two vertices have a common endpoint in G.

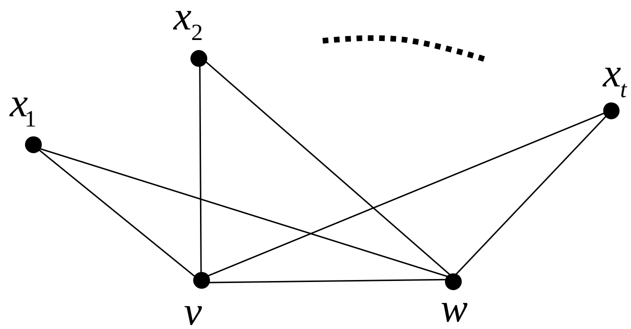

Figure 3: The graph W_t with $t \geq 2$

For each vertex v of G, the set e_v consisting of all edges in G containing v induces a maximal clique in $L(G)$. This is one of the only two types of maximal cliques in $L(G)$, while the rest of maximal cliques is induced by triangles in G. Besides, any edge $ef \in E(L(G))$ with $e = uv$ and $f = vw$ being two edges in G can only be contained in either a clique induced by e, f possibly together with some edges in G with v as endpoint or the clique induced by the triangle uvw in G (if u is adjacent to w). Clearly the set $P = \{e_v : v \in G, d(v) \geq 2\}$ is a clique partition of $L(G)$ which we will call the *canonical clique partition* of $L(G)$. Note that each vertex of $L(G)$ is contained in exactly two cliques in P.

Let G be a graph. A *wing* in G is a triangle with the property that exactly two of its vertices have degree two in G, while a *3-wing* is a wing with the vertex in it having degree greater than two having degree exactly three. Besides, we define a *star* in G to be a collection of edges in G which intersect on a common vertex. Note that a star need not consist of all edges incident with some vertex, but only a sub-collection of those edges. We will use the notation S_v^i to indicate a star with i edges, centered at v. The *join* of simple graphs G and H, denoted $G \vee H$, is the graph obtained from the vertex-disjoint union $G + H$ by adding all the edges $\{xy : x \in V(G), y \in V(H)\}$. We denote the graph by W_t, $t \geq 2$, as in Figure 3.

S. McGuinness and R. Rees proved the following theorem to count the number of distinct minimum edge clique partitions.

Theorem 4.1. (S. McGuinness and R. Rees[10], 1990) *Let G be a connected graph, and $G \neq K_3, K_4, (K_2 + K_2 + K_2) \vee K_1$ (or $3K_2 \vee K_1$ in abbreviation), W_t with $t \geq 2$. Let $V_2(G)$ denote the set of vertices in G with degree at least two , and let w_3 denote the number of 3-wing in G. Then $cp(L(G)) = |V_2(G)|$ and there are exactly 2^{w_3} distinct minimum clique partitions of $L(G)$.*

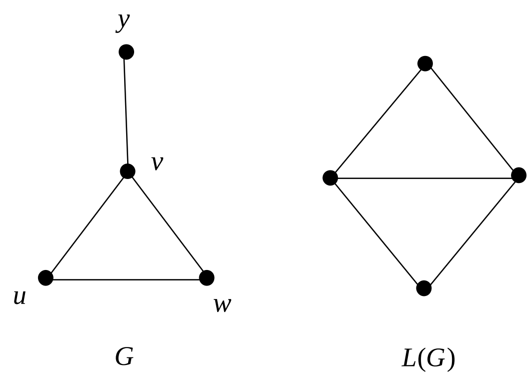

Figure 4: A graph and its line graph

A cursory illustration of the above theorem here would be advantageous for our further study. Note that the above theorem wouldn't concern itself with "isomorphism", that is, it would regard two clique partitions to be distinct if the cliques in two clique partitions are not derived from the same stars and triangles in G. For illustration, to look up a minimum clique partition of the line graph of the graph G in Figure 4, we have two "distinct manners", one by the upper triangle and the inferior two edges in $L(G)$, that is, by the three stars in G centered at u, v, w, whereas the other by the inferior triangle and the upper two edges in $L(G)$, that is, by the 3-wing uvw and the two stars $\{vw, vy\}$, $\{vu, vy\}$ in G.

We will follow this criterion when deciding whether or not two clique partitions are the same. The above theorem clarify the fact that to attain a minimum clique partition of $L(G)$, where note that G is the class of graphs aforementioned in the above theorem, no triangle in $L(G)$ induced by a triangle in G other than 3-wing can be used. And the adopting in a clique partition of $L(G)$ of any triangle induced by one 3-wing in G can also yield a minimum clique partition other than the unique other minimum clique partition, called the canonical one, which consists of all maximal cliques of $L(G)$ induced by one maximal star in G. Thus each 3-wing in G, refer to Figure 4, yield two distinct clique partitions of $L(G)$, one adopting the upper triangle and the inferior two edges in the right graph of Figure 4, while the other adopting the inferior triangle and the upper two edges; and therefore as the aforementioned by the above theorem G has exactly 2^{w_3} distinct minimum clique partitions.

Now we investigate the intersection number of the line graph $L(G)$ of a connected simple graph $G \neq K_3, K_4, 3K_2 \vee K_1$, or $W_t, t \geq 2$.

Theorem 4.2. *Let G be a connected simple graph, and $G \neq K_3, K_4, 3K_2 \vee K_1$, or $W_t, t \geq 2$. In addition, we suppose that G is not a star. Let $V_2(G)$*

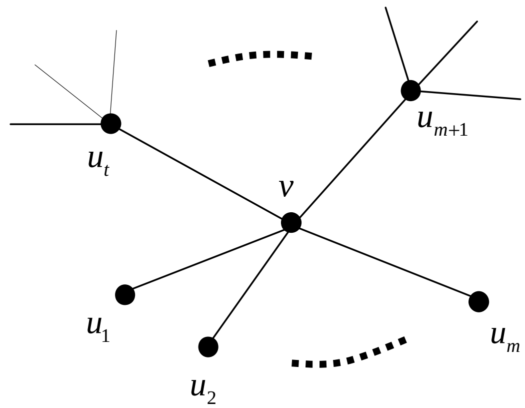

Figure 5: Example

denote the set of vertices in G with degree at least two, and let w_3 denote the number of 3-wing in G. And let $u_1^{(i)}, ..., u_{m_i}^{(i)}$ be all vertices in G of degree one and adjacent to v_i with $d(v_i) > 1$. We suppose that there are k vertices with its degree more than one in G in total which are adjacent to some vertex of degree one, i.e., $1 \leq i \leq k$. Then $\omega_f(L(G)) = |V_2(G)| + \sum_{i=1}^{k}(m_i - 1)$ and there are exactly 2^{w_3} distinct minimum representations of $L(G)$.

Proof. First we consider the following question: When do a minimum clique partition, say $\mathcal{Q}$, of $L(G)$ has two vertices obtaining the same representation set after we take $\mathcal{F}(\mathcal{Q})$? Clearly if such two vertices, say e_1, e_2, exist, then their two corresponding edges in G, say vu_1, vu_2, inetrsect and either $d(u_1) = d(u_2) = 1$ or vu_1u_2 is a wing in G with $d(u_1) = d(u_2) = 2$.

For the former case see Figure 5, where for the sake of generality we suppose that $u_1, ..., u_m$ are vertices in G with degree one and $u_{m+1}, ..., u_t$ with degree at least 2. Immediately after we ask the question whether or not we can represent the complete subgraph K_m in $L(G)$, refer to Figure 5, with vertex set $\{vu_1, ..., vu_m\}$ by exactly m elements in some minimum representation of $L(G)$. (Note that it is impossible to represent it by $m-1$ elements.) Assuming that we can, then this K_m's representation can correspond to three types of clique partitions, say the corresponding clique partition being $\mathcal{Q}$, that is, near-pencil, projective plane, or K_m together with $m-1$ trivial cliques. Note that projective plane and near-pencil have a common property, that is, any two lines intersect on a common point , or paraphrased into terms of clique partition, any two cliques intersect on a common vertex, and recall that in the method by which we construct a correspondence between multifamily representation and clique partition, an element in multifamily representation correspond to a clique in clique partition. Thus for the former two cases, to make $vu_{m+1}, ..., vu_t$ be adjacent to $vu_1, ..., vu_m$, we shouldn't rely on more

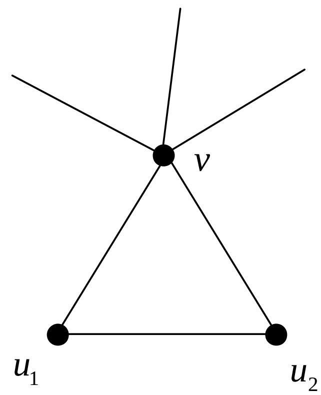

Figure 6: Example

than one element in $\bigcup_{i=1}^m S_{\mathcal{Q}}(vu_i)$, since for any two elements, the vertex on which the two cliques respectively corresponding to them intersect has its representation set comprising them. Nor should we use one. (Unless G itself is a star, that is, $t = m$.) But for the third case we can use the element in $\bigcup_{i=1}^m S_{\mathcal{Q}}(vu_i)$ corresponding to the clique K_m (and note that this is the unique approach if we would like not to use new elements).

On the other hand, for the case that vu_1u_2 is a wing in G with $d(u_1) = d(u_2) = 2$, refer to Figure 6. In this case, whether or not vu_1u_2 is a 3-wing in G, that is, whether or not the adopting of the triangle in $L(G)$ induced by the triangle vu_1u_2 in G can occur in one minimum clique partition of $L(G)$, we can't have $S_{\mathcal{Q}}(vu_1) = S_{\mathcal{Q}}(vu_2)$. $\square$

The case that $G = K_3$ or a star is not hard to see. For $G = 3K_2 \vee K_1$, S. McGuinness and R. Rees [10] have shown that $L(G)$ admits exactly three distinct minimum clique partitions, and with a little direct inspection we see that these three partitions correspond to three distinct minimum (antichain) representations respectively. (As a matter of fact, two of the three are isomorphic.)

As for $G = K_4$, it is easily verified that there are exactly two distinct but in fact isomorphic clique partitions, one by all the cliques in $L(G)$ induced by some maximal star in G, while the other by all the triangles in $L(G)$ induced by some triangle in G, and with a little direct inspection we see that these two partitions correspond to two distinct minimum (antichain) representations respectively.

As for $G = W_t, t \geq 2$, S. McGuinness and R. Rees [10] have shown that $L(G)$ has exactly two distinct minimum clique partitions, and with a little direct inspection we see that these two partitions correspond to two distinct minimum (antichain) representations respectively.

5 Antichain Representations of Line Graphs

Next we consider the antichain intersection number of the line graph $L(G)$, where G is connected simple and $\neq K_3, K_4, 3K_2 \vee K_1$, or $W_t, t \geq 2$.

Theorem 5.1. *Let G be a connected simple graph, and $G \neq K_3, K_4, 3K_2 \vee K_1$, or $W_t, t \geq 2$. In addition, we suppose that G is not a star, and is not a graph as in Figure 11, 12, 13, 14. Let $V_2(G)$ denote the set of vertices in G with degree at least two, and let w_3 denote the number of 3-wing in G. And let $u_1^{(i)}, ..., u_{m_i}^{(i)}$ be all vertices in G of degree one and adjacent to v_i with $d(v_i) > 1$. We suppose that there are k vertices with its degree more than one in G in total which are adjacent to some vertex of degree one, i.e., $1 \leq i \leq k$. And we suppose that there are altogether k' such numbers i in $\{1, ..., k\}$ so that $t_i = m_i + 1$, and that among the k' numbers there are k'' such numbers i so that there exists projective plane with t_i vertices. Then when regarding $L(G)$ as a multigraph, $\omega_a(L(G)) = |V_2(G)| + \sum_{i=1}^{k} m_i$ and there are exactly $3^{k'-k''}4^{k''}$ distinct minimum antichain representations of $L(G)$.*

Proof. First, we consider the question that when do a minimum clique partition, say $\mathcal{Q}$, of $L(G)$ has two vertices the two corresponding representation sets for which after we take $\mathcal{F}(\mathcal{Q})$ would have one contained in the other. Clearly, the two edges in G, say e_1, e_2, corresponding to such two vertices must intersect, say $e_1 = vu_1, e_2 = vu_2$, and one of u_1, u_2, say u_1 throughout the rest of this paper, has no neighbor other than v, u_2.

We first consider exclusively the case that vu_1u_2 form a triangle in G. Now $d(u_1) = 2$. If only we have never made use of the clique in $L(G)$ induced by the triangle vu_1u_2 in G in a clique partition, say $\mathcal{Q}$, of $L(G)$, we utterly needn't to worry about the inclusion relation between the two representation sets $S_\mathcal{Q}(e_1), S_\mathcal{Q}(e_2)$. Thus what we need to consider is mere the case that there exists a minimum clique partition of $L(G)$ making use of the triangle in $L(G)$ induced by the triangle vu_1u_2 in G, i.e., that the triangle vu_1u_2 is a 3-wing. Recall that we have supposed that $d(u_1) = 2$, and thus exactly one of v, u_2 has degree two and the other has degree three. In case that $d(v) = 2$, making use of the triangle vu_1u_2 in a minimum clique partition, say $\mathcal{Q}$, will make $S_\mathcal{Q}(e_1)$ be contained in $S_\mathcal{Q}(e_2)$. Thus in this case the representation derived from the minimum clique partition of $L(G)$ making no use of the triangle vu_1u_2, i.e., the canonical one, is the unique approach to form an minimum antichain representation of $L(G)$. In case that $d(u_2) = 2$, whether or not we make use of the triangle vu_1u_2 in a minimum clique partition, say $\mathcal{Q}$, of $L(G)$, there can't be inclusion relation between $S_\mathcal{Q}(e_1), S_\mathcal{Q}(e_2)$. But if we make use of the triangle vu_1u_2, then $S_\mathcal{Q}(u_1u_2)$ will be contained in both

$S_{\mathcal{Q}}(e_1)$ and $S_{\mathcal{Q}}(e_2)$. Thus in this case we have the same conclusions as the former one.

Now what remained is the case that u_1 is not adjacent to u_2. For this case, we can without loss of generality assume that $d(u_1) = 1$ while leave $d(u_2)$ unappointed. See Figure 5, where for the sake of generality we suppose that $u_1, ..., u_m$ are vertices in G with degree one and $u_{m+1}, ..., u_t$ with degree at least two. Immediately after we look for a minimum antichain representation of $L(G)$ in which the complete subgraph K_m with vertex set $vu_1, ..., vu_m$ is represented using exactly m elements. (Note that it is impossible to represent it by $m-1$ elements.) Assuming that we can, then this K_m's representation can only correspond to two types of clique partitions, say the corresponding clique partition being $\mathcal{Q}$, that is, near-pencil or projective plane. (When $m = 1$, we can represent K_m by m elements with respect to antichain. But in this case we can't make u_1 be adjacent to $u_{m+1}, ..., u_t$ by the single element in the representation set of u_1 so that the representation set of u_1 wouldn't be contained in the representation sets of $u_{m+1}, ..., u_t$, unless $t = 1$, that is, $G = K_2$.) Now to make $vu_{m+1}, ..., vu_t$ be adjacent to $vu_1, ..., vu_m$, we can't use more than one element in $\bigcup_{i=1}^{m} S_{\mathcal{Q}}(vu_i)$ for securing the representation sets of any two vertices from overlapping on more than one element, neither can we use one (unless G itself is a star, that is, $t = m$).

Thus we should yield by one step looking for a minimum antichain representation of $L(G)$ in which the complete subgraph K_m with vertex set $\{vu_1, ..., vu_m\}$ is represented by exactly $m + 1$ elements. Assuming such a minimum antichain representation, then by theorem 2.1 and 3.3 this K_m's representation can only correspond to five types of clique partitions, say the corresponding clique partition being $\mathcal{Q}$, that is, near-pencil together with one trivial clique attached on it, projective plane together with one trivial clique attached on it, one K_m together with m trivial cliques attached on it, one as in Figure 2, or projective plane with one vertex deleted.

For the first case, to make $vu_{m+1}, ..., vu_t$ be adjacent to $vu_1, ..., vu_m$, we can't use more than one element in $\bigcup_{i=1}^{m} S_{\mathcal{Q}}(vu_i)$ different from the monopolized one for securing the representation sets of any two vertices from overlapping on more than one element. Since we have only one monopolized element in $\bigcup_{i=1}^{m} S_{\mathcal{Q}}(vu_i)$, thus we must try to use one non-monopolized element in $\bigcup_{i=1}^{m} S_{\mathcal{Q}}(vu_i)$ to make $m - 1$ vertices of the K_m be adjacent to $vu_{m+1}, ..., vu_t$ (unless G is a star, i.e., $t = m$), and then use the monopolized element on the vertex of the K_m other than the aforementioned $m - 1$ vertices to make this vertex be adjacent to $vu_{m+1}, ..., vu_t$. Clearly we have only one approach to do so, that is, first take the element in $\bigcup_{i=1}^{m} S_{\mathcal{Q}}(vu_i)$ that correspond to the clique K_{m-1} in $\mathcal{Q}$ to make all vertices on this K_{m-1} be adjacent to $vu_{m+1}, ..., vu_t$, and then use the monopolized element on the

vertex not on this K_{m-1} to make this vertex be adjacent to $vu_{m+1}, ..., vu_t$. But when $t > m+1$, using this method will make $|S(u_{m+1}) \cap S(u_{m+2})| \geq 2$. Thus, provided that G is not a star, this method can be carried out only if $t = m+1$.

As for the second case, i.e. projective plane together with one trivial clique attached on it, similarly we must try to use one non-monopolized element in $\bigcup_{i=1}^m S_{\mathcal{Q}}(vu_i)$ to make $m-1$ vertices of the K_m be adjacent to $vu_{m+1}, ..., vu_t$ (unless G is a star, i.e., $t = m$). But we know that in a projective plane of order k each clique contain $k+1$ vertices, whereas there are k^2+k+1 vertices in total where $k \geq 2$, and thus each clique in a projective plane has

$$(k^2 + k + 1) - (k + 1) = k^2 \geq 4$$

vertices not on it. Thus in this case we have failed.

For the third case, i.e., one clique K_m together with m trivial cliques attached on it, for the sake not to make two representation sets overlap on more than one element, we can only use the element in $\bigcup_{i=1}^m S_{\mathcal{Q}}(vu_i)$ corresponding to the clique K_m in $\mathcal{Q}$ or all monopolized elements in $\bigcup_{i=1}^m S_{\mathcal{Q}}(vu_i)$ to make $vu_{m+1}, ..., vu_t$ be adjacent to $vu_1, ..., vu_m$. But when $t > m+1$ and there is one vertex, say u_{m+1}, in $\{u_{m+1}, ..., u_t\}$ which is not adjacent to any other vertex in $\{u_{m+1}, ..., u_t\}$, then for the sake that we should make vu_{m+1} be adjacent to $vu_{m+2}, ..., vu_t$, we can only use the element in $\bigcup_{i=1}^m S_{\mathcal{Q}}(vu_i)$ corresponding to this K_m for vu_{m+1} to be adjacent to $vu_1, ..., vu_m$. (If we use the monopolized elements corresponding to all trivial cliques in $\mathcal{Q}$ for vu_{m+1} to be adjacent to $vu_1, ..., vu_m$, then since there is no triangle in G which contains v and u_{m+1} by our supposition before, so in any clique partition of $L(G)$ we can only cover the edge $\{vu_{m+1}, vu_{m+2}\}$ by a clique induced by some star in G centered at v. Thus we can use neither the element in $\bigcup_{i=1}^m S_{\mathcal{Q}}(vu_i)$ corresponding to K_m nor all monopolized elements in $\bigcup_{i=1}^m S_{\mathcal{Q}}(vu_i)$ for vu_{m+2}, or otherwise either we can't make vu_{m+2} be adjacent to vu_{m+1} or we will make the representation sets of vu_{m+1}, vu_{m+2} overlap on more than one element.) Thus in this case, when $t > m+1$ we have only one method to make a vertex belonging to $vu_{m+1}, ..., vu_t$ but not adjacent to any member of it be adjacent to $vu_1, ..., vu_m$ using elements in $\bigcup_{i=1}^m S_{\mathcal{Q}}(vu_i)$, while when $t = m+1$ we have two methods to make vu_{m+1} be adjacent to $vu_1, ..., vu_m$ using elements in $\bigcup_{i=1}^m S_{\mathcal{Q}}(vu_i)$.

As for the forth case, i.e. one as in Figure 2, for securing the representation sets of any two vertices from overlapping on more than one element, we need one pair of vertex-disjoint cliques in the clique partition as in Figure 2, and the unique two vertex-disjoint pairs of cliques, refer to Figure 2, are $\{Q_3, Q_4\}$ and $\{Q_5, Q_6\}$. If we use Q_3, Q_4 to make $vu_{m+1}, ..., vu_t$ be adjacent

to v_2, v_3, v_4, v_5, then to make v_1 be adjacent to $vu_{m+1}, ..., vu_t$ we can use neither 1 nor 2 for the sake of two representation sets overlapping on more than one element. Similarly for the use of Q_5, Q_6. Thus in this case we have failed.

For the fifth case, i.e. projective plane, say of order $k \geq 2$, with one vertex, say x, deleted, we know that a clique in this clique partition has at most $k+1$ vertices, whereas there are $k^2 + k$ vertices in total. Thus in this case each clique has at least

$$(k^2 + k) - (k + 1) = k^2 - 1 \geq 3$$

vertices not on it. Thus in order that $vu_{m+1}, ..., vu_t$ be adjacent to $vu_1, ..., vu_m$, we need more than one element from $\mathcal{F}(\mathcal{Q})$. Recall that a projective plane with k^2+k+1 points for some $k \geq 2$ has point and line regularity $k+1$. Thus deleting one vertex from a projective plane of order $k \geq 2$ leaves a clique partition consisting of $k+1$ cliques of cardinality k and k^2 cliques of cardinality $k+1$. Besides, recall that any two cliques in a projective plane intersect on a common vertex. Thus we couldn't adopt two elements in $\bigcup_{i=1}^{m} S_{\mathcal{Q}}(vu_i)$ which correspond to two cliques in $\mathcal{Q}$ of cardinality $k+1$ to make $vu_{m+1}, ..., vu_t$ be adjacent to $vu_1, ..., vu_m$, or otherwise the representation set (turned out after we take $\mathcal{F}(\mathcal{Q})$) of the vertex on which the two cliques intersect and the representation sets of $vu_{m+1}, ..., vu_t$ would overlap on more than one element (unless $t = m$, that is, G is a star). Nor could we adopt two elements in $\bigcup_{i=1}^{m} S_{\mathcal{Q}}(vu_i)$ corresponding to two cliques in $\mathcal{Q}$ respectively of cardinality $k, k+1$, for the same reason. Now the only permissible choice is the adoption of elements in $\bigcup_{i=1}^{m} S_{\mathcal{Q}}(vu_i)$ corresponding to the $k+1$ cliques in $\mathcal{Q}$ of cardinality k. The vertex, say x, on which these $k+1$ cliques would intersect but for the deletion of x from the primitive projective plane of order k, having been deleted, these $k+1$ cliques are pairwisely vertex-disjoint. (Recall the property of one linear space that any two lines intersect on at most one point.) There are altogether $k(k+1) = k^2 + k$ vertices in these $k+1$ cliques, tantamount to the sum total of vertices in $\mathcal{Q}$. Thus we could utilize the $k+1$ elements corresponding to these $k+1$ cliques in order that $vu_{m+1}, ..., vu_t$ be adjacent to $vu_1, ..., vu_m$. Note that this method can be carried out only when $t = m + 1$ for securing two vertices from having their representation sets overlapping on more than one element.

To summerize we have obtained the following lemma.

Lemma 5.2. *Let G be a connected simple graph, and $G \neq K_3, K_4, 3K_2 \vee K_1$, or $W_t, t \geq 2$. In addition, we suppose that G is not a star. And let $u_1^{(i)}, ..., u_{m_i}^{(i)}$ be all vertices in G of degree one and adjacent to v_i with $d(v_i) > 1$, while $u_{m_i+1}^{(i)}, ..., u_{t_i}^{(i)}$ be all vertices in G of degree more than one and adjacent*

to v_i. We suppose that there are k vertices with its degree more than one in G in total which are adjacent to some vertex of degree one, i.e., $1 \leq i \leq k$.

Then for any $1 \leq i \leq k$ so that $t_i = m_i + 1$, we have exactly four distinct minimum antichain representations of $L(G)$ respectively corresponding to four distinct methods for representing the clique of $L(G)$ with vertex set $\{v_i u_1^{(i)}, ..., v_i u_{t_i}^{(i)}\}$. Figure 7 illustrates these four distinct methods where for illustration we suppose that $m_i = 4$ in the upper three graphs and that $v_i u_1^{(i)}, ..., v_i u_6^{(i)}$ form a projective plane of order 2 with one vertex deleted in the lowermost graph. Note that the method corresponding to the lowermost graph of Figure 7 relies on the existence of projective plane with t_i vertices.

On the other hand, for any $1 \leq i \leq k$ so that $t_i > m_i + 1$, all minimum antichain representations of $L(G)$ have the same method for representing the clique of $L(G)$ with vertex set $\{v_i u_1^{(i)}, ..., v_i u_{m_i}^{(i)}\}$, and for any vertex in $\{v_i u_{m_i+1}^{(i)}, ..., v_i u_{t_i}^{(i)}\}$ which is not adjacent to any other member of it, all minimum antichain representations of $L(G)$ also have the same method to make this vertex be adjacent to $v_i u_1^{(i)}, ..., v_i u_{m_i}^{(i)}$ using elements in $\bigcup_{j=1}^{m_i} S(v_i u_j^{(i)})$. Figure 8 illustrate this unique method, where for illustration we suppose that $m_i = 4$ and $v_i u_{m_i+1}^{(i)}$ is a such vertex, which is not adjacent to any other member of $\{v_i u_{m_i+1}^{(i)}, ..., v_i u_{t_i}^{(i)}\}$. For any vertex in $\{v_i u_{m_i+1}^{(i)}, ..., v_i u_{t_i}^{(i)}\}$ which is adjacent to some other member of it, a minimum antichain representation of $L(G)$ would make this vertex be adjacent to $v_i u_1^{(i)}, ..., v_i u_{m_i}^{(i)}$ using elements in $\bigcup_{j=1}^{m_i} S(v_i u_j^{(i)})$ by one of the two methods as described in Figure 9, where for illustration we suppose that $m_i = 4$ and $v_i u_{m_i+1}^{(i)}$ is a such vertex, which is adjacent to $v_i u_{m_i+2}^{(i)}$.

Note that once a vertex in $\{v u_{m_i+1}^{(i)}, ..., v u_{t_i}^{(i)}\}$ adopts the representation method as the right in Figure 9, then all other vertices in $\{v u_{m_i+1}^{(i)}, ..., v u_{t_i}^{(i)}\}$ must all adopt the representation method as the left in Figure 9, or otherwise there will be two vertices in $\{v u_{m_i+1}^{(i)}, ..., v u_{t_i}^{(i)}\}$ overlapping on more than one element on their representation sets.

Due to the above lemma, what is still vague is mere the case that $d(u_1) = 1$ and there is some triangle on v in G, see Figure 10 where for illustration we suppose that $u_1, ..., u_m$ are the all vertices in G adjacent to v and with degree one, u_{m+1} is a vertex adjacent to v and with degree at least two so that there is no triangle in G containing the edge vu_{m+1}, and v, u_{m+2}, u_{m+3} form a triangle in G.

By Lemma 5.2, if only we can prove that using the method as the left in Figure 9 is always not worst than the one as the right in Figure 9 in sense of the intent to minimize a representation of $L(G)$, where G is connected,

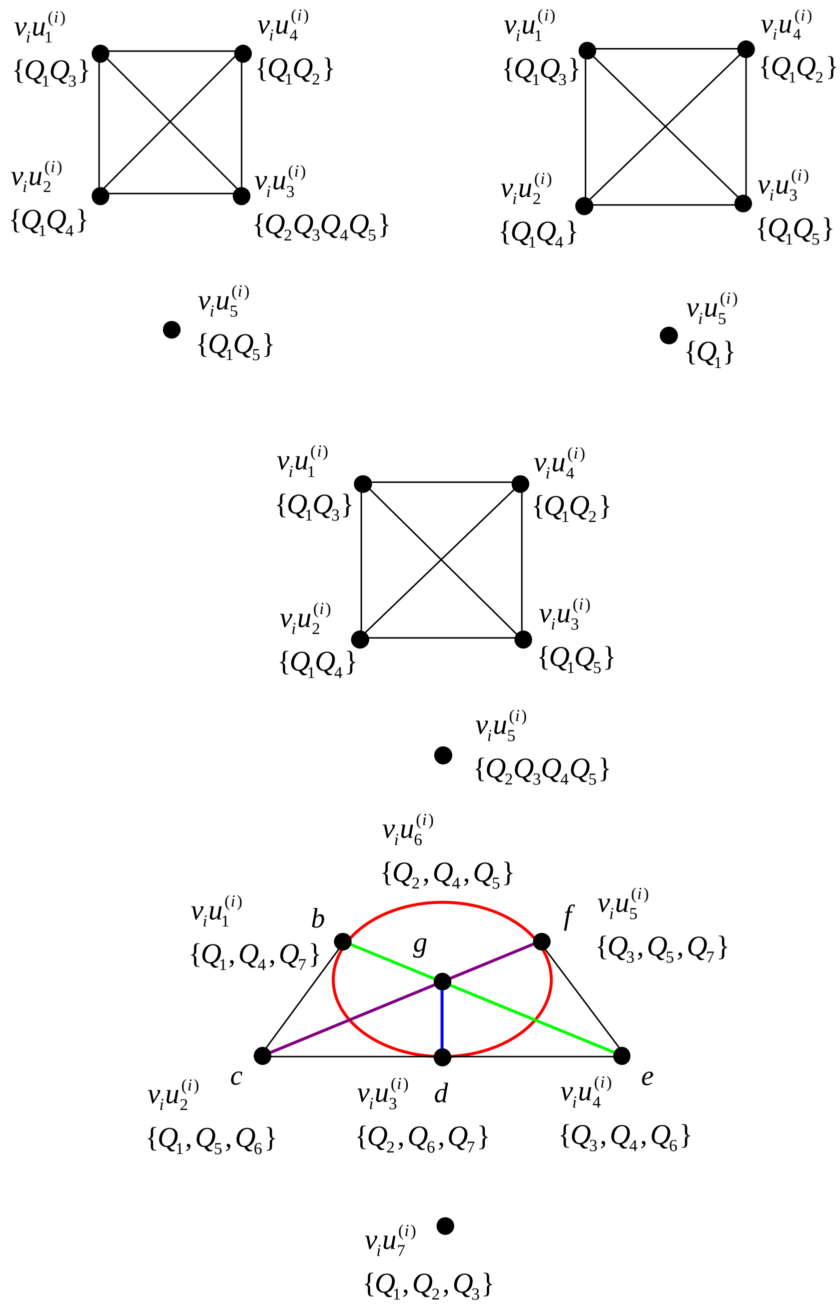

Figure 7: Example

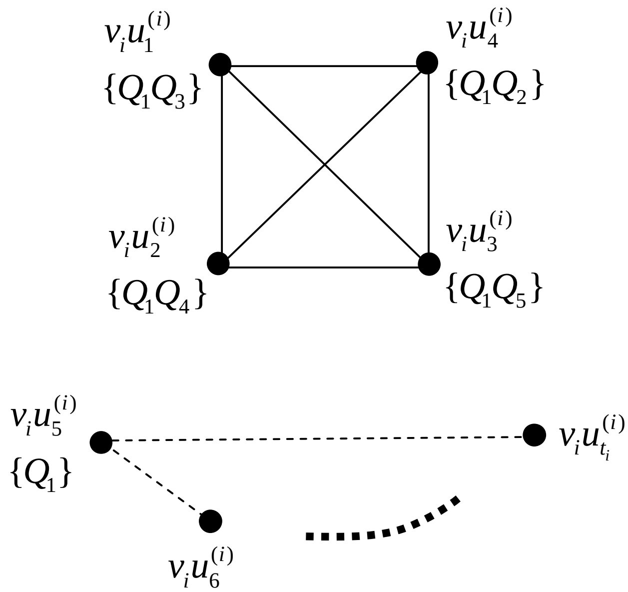

Figure 8: Example

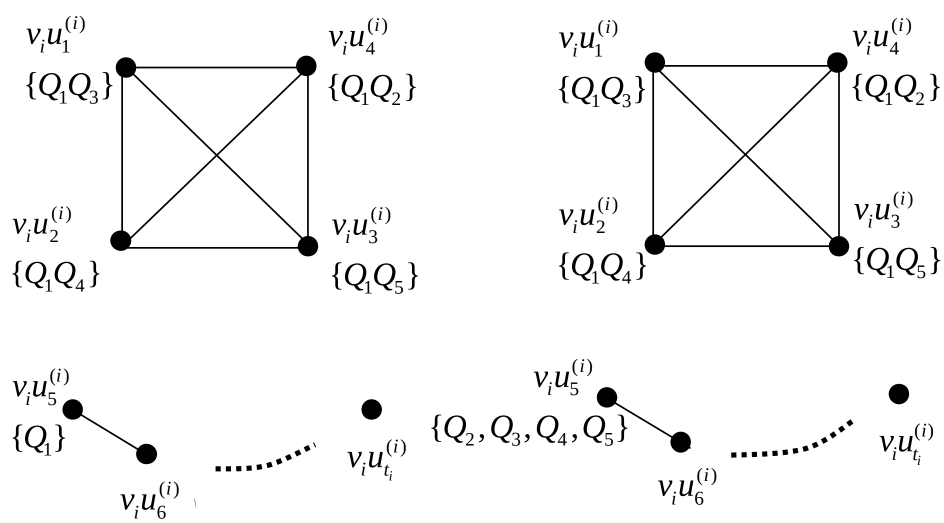

Figure 9: Example

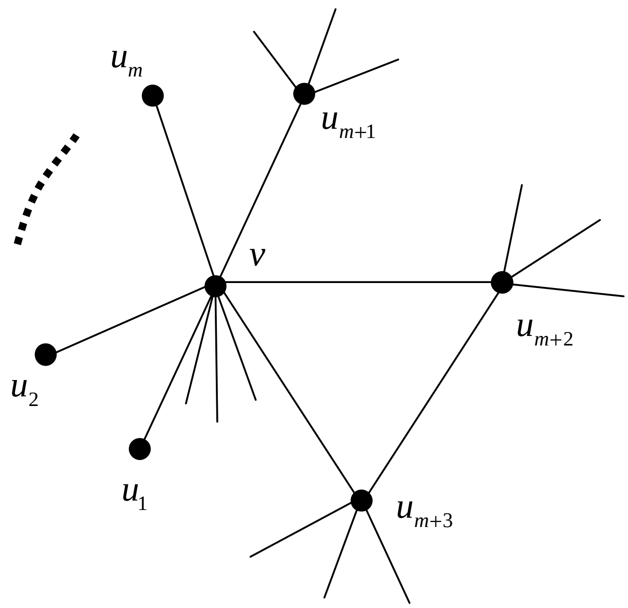

Figure 10: Example

$\neq K_3, K_4, 3K_2 \vee K_1,$ or $W_t, t \geq 2$ and is not a star, and characterize all situations under which the two methods in Figure 9 is equally fine, then we can determine the antichain intersection number of any line graph and whether or not any line graph is uniquely intersectable with respect to antichain.

We examine the method as the left in Figure 9. In this method, refer to Figure 10, we use one element to make the vertices $vu_1, ..., vu_t$, where we say that $d(v) = t$, be adjacent to each other, and use m monopolized elements respectively in the representation sets of $vu_1, ..., vu_m$. Thus in the whole $L(G)$, we use $|V_2(G)| + \sum_{i=1}^{k} m_i$ elements, where $V_2(G)$ denote the set of vertices of degree at least two in G and we let $v_i, 1 \leq i \leq k$ be all vertices of degree more than one in G which is adjacent to some vertex of degree one and for $1 \leq i \leq k$, $u_1^{(i)}, ..., u_{m_i}^{(i)}$ be all vertices in G of degree one and adjacent to v_i.

Immediately after we examine the method as the right in Figure 9. In this method, refer to Figure 10, we use m elements to make vu_{m+2} be adjacent to $vu_1, ..., vu_m$, respectively; and use one more element to make all u_i with $1 \leq i \leq t$ and $i \neq m+2$ be adjacent to each other. Note that now we have made use of $m+1$ elements, that is exactly equal to the number of elements we should have used for the v-star if we had adopted the left method in Figure 9. But now we should use still another element to make vu_{m+2} be adjacent to vu_{m+1} (unless u_{m+1} is adjacent to u_{m+2} and thus we can shake off the responsibility to make vu_{m+2} be adjacent to vu_{m+1} to the triangle

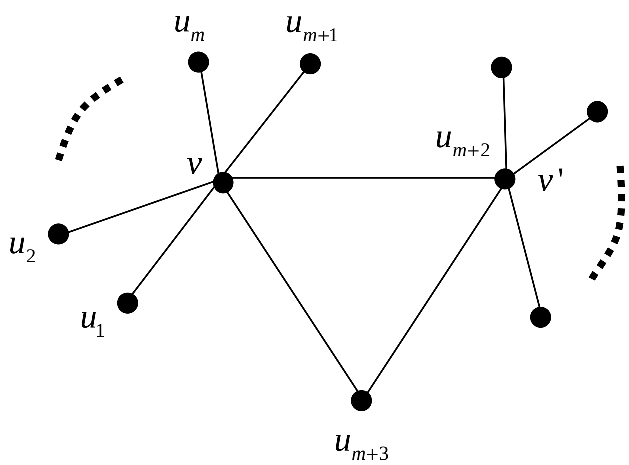

Figure 11: Example

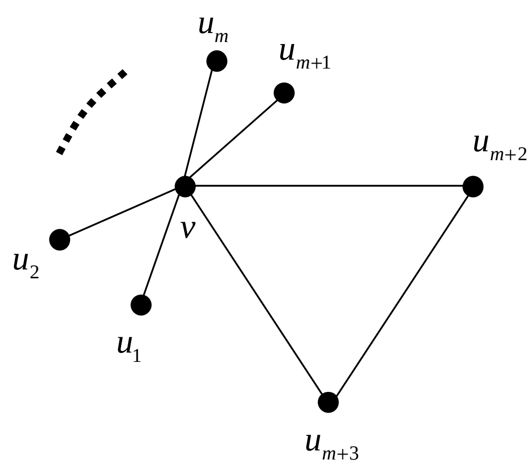

Figure 12: Example

$\{vu_{m+1}, vu_{m+2}, u_{m+1}u_{m+2}\}$ just like how we will deal with the responsibility to make vu_{m+2} be adjacent to vu_{m+3}). But even if u_{m+1} is adjacent to u_{m+2}, where to dispose of the triangle $\{vu_{m+1}, vu_{m+2}, u_{m+1}u_{m+2}\}$? If only $d(u_{m+1}) = 2$, we can shake off this triangle to the star $\{vu_{m+1}, u_{m+1}u_{m+2}\}$. Thus to attain a minimum antichain representation we should have either that u_{m+1} is adjacent to u_{m+2} and $d(u_{m+1}) = d(u_{m+3}) = 2$, or that $d(u_{m+1}) = 1$ and $d(u_{m+3}) = 2$. For the latter case, see Figure 11, where note that by symmetry we also have all neighbors of u_{m+2} being of degree one. When $d(u_{m+1}) = 1$ and $d(u_{m+2}) = d(u_{m+3}) = 2$, see Figure 12. As for the case that u_{m+1} is adjacent to u_{m+2} and $d(u_{m+1}) = d(u_{m+3}) = 2$, see Figure 13. There is another case left, see Figure 14. Therefore we complete the proof. $\square$

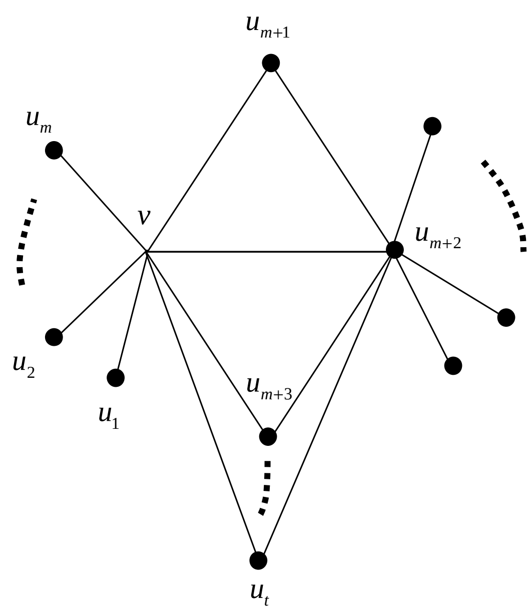

Figure 13: Example

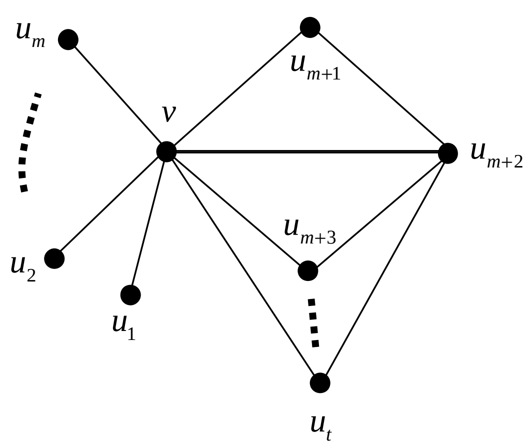

Figure 14: Example

In Figure 11, if we use the triangle $\{vu_{m+2}, vu_{m+3}, u_{m+2}u_{m+3}\}$, i.e., use one element to make the three vertices $vu_{m+2}, vu_{m+3}, u_{m+2}u_{m+3}$ be adjacent to each other, and either use one more element to make all vu_i with $1 \leq i \leq m+3$ and $i \neq m+2$ be adjacent to each other and $m+1$ more elements to respectively make vu_{m+2} be adjacent to $vu_1, ..., vu_{m+1}$ or exchange the roles of u_{m+2} and u_{m+3} and do the same as before, and then similarly for the v'-star, then we can obtain four more distinct minimum antichain representations different from the "canonical one".

In Figure 12, if we use the triangle $\{vu_{m+2}, vu_{m+3}, u_{m+2}u_{m+3}\}$, and use one more element to make all vu_i with $1 \leq i \leq m+3, i \neq m+2$ be adjacent to each other, and use $m+1$ more elements to make vu_{m+2} be adjacent to $vu_1, ..., vu_{m+1}$, respectively, and then attach one monopolized element to the representation set of $u_{m+2}u_{m+3}$ then we obtain one more minimum antichain representation other than "the canonical one".

In Figure 13, if we use the $t-m$ triangles

$$\{vu_{m+1}, vu_{m+2}, u_{m+1}u_{m+2}\}, \{vu_{m+2}, vu_{m+3}, u_{m+2}u_{m+3}\}, \\ ..., \{vu_{m+2}, vu_t, u_{m+2}u_t\},$$

and use one more element to make all vu_i with $1 \leq i \leq t, i \neq m+2$ be adjacent to each other, and use m more elements to make vu_{m+2} be adjacent to $vu_1, ..., vu_m$, respectively, and then do the same for the u_{m+2}-star, we will obtain one more minimum antichain representation other than "the canonical one".

In Figure 14, if we use the $t-m$ triangles

$$\{vu_{m+1}, vu_{m+2}, u_{m+1}u_{m+2}\}, \{vu_{m+2}, vu_{m+3}, u_{m+2}u_{m+3}\}, \\ ..., \{vu_{m+2}, vu_t, u_{m+2}u_t\},$$

and use one more element to make all vu_i with $1 \leq i \leq t, i \neq m+2$ be adjacent to each other, and then use one more element to make all $u_{m+2}u_i$ with $m+1 \leq i \leq t$ and $i \neq m+2$ be adjacent to each other, we will obtain one more minimum antichain representation other than "the canonical one".

6 Conclusion Remarks

Edge clique partitions, as a special case of edge clique covers, are served as great classifying and clustering tools in many practical applications, therefore it is interesting to explore the concept in more details.

One may keep working on the set representations of graphs in various

senses. Also the relationships among these various representations are interesting to be explored further.

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